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On a q-generalization of circular and hyperbolic functions

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Abstract. A generalization of the circular and hyperbolic functions is proposed, based on the Tsallis statistics and on a consistent generalization of the Euler formula. Some properties of the presently proposed q-trigonometry are then investigated. The generalized functions are exact solutions of a nonlinear oscillator. Original circular and hyperbolic relations are recovered as the $q \to 1$ limiting case.

1. Introduction

The q-analysis began at the end of the 19th century, as stated by McAnally [1], recalling the work of Rogers [2] on the expansion of infinite products. Recently, however, its use and importance has increased, owing to its relationship with quantum groups [3], and its development brought together the need for the generalization of special functions to handle nonlinear phenomena [4]. The problem of the q-oscillator algebra [5], for example, has led to q-analogues of many special functions, in particular the q-exponential and the q-gamma functions [1, 6], the q-trigonometric functions [7], q-Hermite and q-Laguerre polynomials [3, 8], which are particular cases of q-hypergeometric series.

The q-exponential, for example, is defined by [1, 9] $\mathrm{e}_q(x) = \sum_n x^n/(n)_q!$, with $(n)_q! = \prod_{j=1}^{n}(j)_q$ and $(j)_q = (q^j - 1)/(q-1)$ and also $(0)_q! = 1$. In this paper we shall explore a *different* q-deformation of the exponential function, that emerges from Tsallis statistics.

Recently a connection between quantum groups and statistical mechanics has been proposed by Tsallis [10–12] through the concept of a generalized entropy defined by [13] $S_q \equiv k(1 - \sum_{i=1}^{W} p_i^q)/(q-1)$, $(q \in \mathcal{R})$, where $\{p_i\}$ are the probabilities associated with W microstates (configurations), k is a positive constant and q is the parameter that generalizes the statistics. If q is set to unity, the usual Boltzmann expression is recovered: $S_1 = -k\sum_{i=1}^{W} p_i \ln p_i$.

Tsallis statistics has been shown to preserve the Legendre transformation structure of thermodynamics [14], and also to satisfy generalized forms of the Ehrenfest theorem [15], von Neumann equation [16], H-theorem [17], among others. It has been applied to Lévy [18] and correlated [19] anomalous diffusions, self-gravitating systems [20], turbulence in pure electron plasma [21], cosmology and cosmic background radiation [22], solar neutrinos [23], linear response theory [24], phonon–electron interactions [25], peculiar velocities of

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galaxies [26], nonlinear dynamical systems [27], with promising results. For an extensive and up-to-date bibliography, see [28].

For the microcanonical ensemble, Tsallis entropy is given by [13]

$$S_q = k\frac{W^{1-q}-1}{1-q}. \tag{1}$$

In the $q \to 1$ limit, the q-entropy goes to $S_1 = k\ln W$. The distribution law for the canonical ensemble in the Tsallis formalism is proportional to

$$p_i \propto [1-(1-q)\beta E_i]^{1/(1-q)} \tag{2}$$

where β is the Lagrange parameter and $\{E_i\}$ is the energy spectrum. Equation (2) is reduced to the usual Boltzmann distribution law, $p_i \propto \mathrm{e}_1^{-\beta E_i}$, as $q \to 1$. Note that equations (1) and (2) suggest a form to introduce a q-logarithm and a q-exponential function by defining [29]

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \qquad \exp_q x \equiv \mathrm{e}_q^x = [1+(1-q)x]^{1/(1-q)}. \tag{3}$$

It is immediately verified that $\ln_q x$ and e_q^x are inverse to each other. The ordinary logarithm and exponential functions (here known as $\ln_1 x$ and $\exp_1 x$, or e_1^x) are recovered when $q \to 1$.

Here we are mainly concerned with the study of the q-circular and q-hyperbolic functions that the definitions given in equation (3) lead to. As a result, we show that some such functions, introduced in this context of the Tsallis entropy, are solutions of a nonlinear wave equation. Beyond that, q-generalizations of the Euler formula, Pythagoras theorem and De Moivre theorem are deduced, as well as the roots of the q-sine and q-cosine functions and the relation between q-circular and q-hyperbolic functions.

This paper is organized as follows. In section 2 we introduce the q-circular functions, and establish some of its properties. In section 3 we extend this generalization to the hyperbolic functions, and, finally, in section 4 we state the conclusions and final remarks.

2. Generalized q-circular functions

If we expand $\exp_q x$ in Taylor series around $x_0 = 0$, we find

$$\exp_q x = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} Q_{n-1} x^n \tag{4}$$

with

$$Q_n(q) \equiv 1 \cdot q(2q-1)(3q-2)\ldots[nq-(n-1)]. \tag{5}$$

The q-exponential of an imaginary number ix leads to an expression that reminds us of the Euler formula in complex analysis and we may write

$$\exp_q(\pm \mathrm{i}x) = \cos_q x \pm \mathrm{i}\sin_q x \tag{6}$$

where $\cos_q x$ and $\sin_q x$ represent the generalized q-cosine and q-sine functions, defined by

$$\cos_q x \equiv 1 + \sum_{j=1}^{\infty} \frac{(-1)^j Q_{2j-1} x^{2j}}{(2j)!} \qquad \sin_q x \equiv \sum_{j=0}^{\infty} \frac{(-1)^j Q_{2j} x^{2j+1}}{(2j+1)!}. \tag{7}$$

In the following we are going to show that $\cos_q x$ and $\sin_q x$ satisfy general forms of the usual trigonometric relations. The ratio test shows that equations (4) and (7) converge absolutely within the region $|x| < |1-q|^{-1}$. In the $q \to 1$ limit, $Q_n(1) = 1$, $\forall n \in \mathcal{N}$ and these equations turn to the Taylor expansions of the ordinary exponential, cosine and sine functions, converging for $-\infty < x < \infty$. If we take the q-exponential written as

$$\exp_q x = \exp_1\left[\frac{\ln_1[1+(1-q)x]}{1-q}\right] \qquad \forall x \neq \frac{1}{q-1} \tag{8}$$

and use the property of the 1-logarithm of a complex number $z = |z|e_1^{i\phi}$, namely $\ln_1 z = \ln_1 |z| + i\phi$, we find

$$\cos_q x = \rho_q(x)\cos_1[\varphi_q(x)] \qquad \sin_q x = \rho_q(x)\sin_1[\varphi_q(x)] \tag{9}$$

where

$$\rho_q(x) = \{\exp_q[(1-q)x^2]\}^{1/2} \qquad \varphi_q(x) = \frac{\arctan_1[(1-q)x]}{1-q}\,. \tag{10}$$

We also have

$$\tan_q x = \tan_1[\varphi_q(x)] \tag{11}$$

where the generalized q-tangent is defined as expected,

$$\tan_q x \equiv \frac{\sin_q x}{\cos_q x}\,. \tag{12}$$

According to our notation, $\cos_1 x$, $\sin_1 x$, and $\tan_1 x$ are the usual cosine, sine and tangent functions. Equations (9)–(11) are interesting because they allow q-cosines, q-sines and q-tangents to be expressed in terms of known functions. The q-cosine and q-sine are composed by the product of two factors. The first, $\rho_q(x)$, is responsible for the amplitude, and the second is responsible for the oscillatory nature of these functions. In particular, observe that the q-sine function presents

$$\lim_{x\to 0}\frac{\sin_q x}{x} = 1 \qquad \forall q \in \mathcal{R}. \tag{13}$$

The behaviour of $\cos_q x$ and $\sin_q x$ for different values of $q > 1$ and $q < 1$ are illustrated by figures 1 and 2.

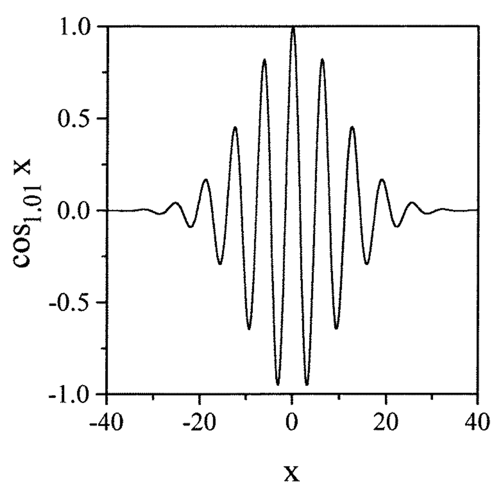

Figure 1. $\cos_{1.01} x$.

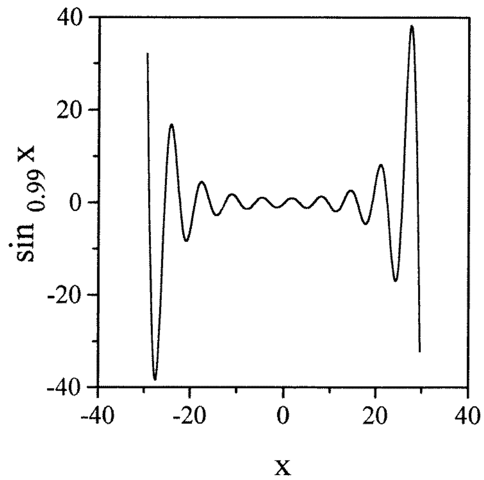

Figure 2. $\sin_{0.99} x$.

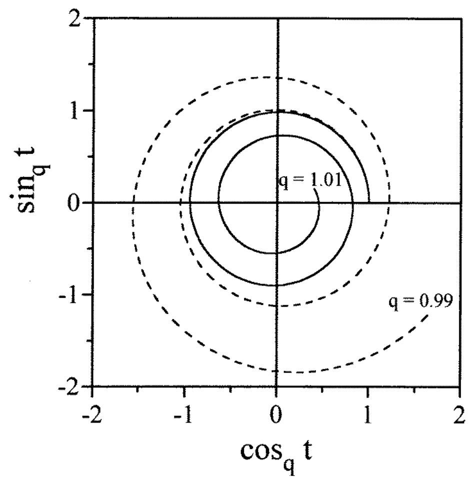

Figure 3. Spiral diagrams for $q = 1.01$ (continuous curve) and $q = 0.99$ (broken curve).

The parametric representation of the q-cosine and q-sine ($x = \cos_q t$, $y = \sin_q t$, $z = t$) represents a helix. Figure 3 shows the projection of the helix on the xy-plane, as viewed from the positive z-side, for different values of q. The spirals go to zero for $q > 1$ and diverge for $q < 1$. If $q \to 1$ the spiral degenerates into a circle (the usual circular functions). The modulus of the radius vector of a point t on the spiral is given by

$$\cos_q^2 t + \sin_q^2 t = \exp_q(\mathrm{i}t)\exp_q(-\mathrm{i}t) = \rho_q^2(t) \tag{14}$$

that is the generalized Pythagoras theorem. These features keep a close analogy with the usual trigonometric circle and suggest that we refer to them as *q-spiral functions*. The number of rotations of these spiral diagrams is *finite*, owing to the fact that there is an absolute maximum value for $\varphi_q(t)$,

$$\varphi_q^{\max} = \lim_{t\to\infty} \varphi_q(t) = \frac{\pi}{2}\left|\frac{1}{1-q}\right| \tag{15}$$

so that $\cos_q t$ and $\sin_q t$ oscillate indefinitely only if $q = 1$. The number of roots of the q-cosine (N_c) and that of the q-sine (N_s) are found to be

$$N_c = 2\left[\operatorname{int}\left(\left|\frac{1}{1-q}\right|\right) - \operatorname{int}\left(\frac{1}{2}\left|\frac{1}{1-q}\right|\right)\right] \qquad N_s = 2\ \operatorname{int}\left(\frac{1}{2}\left|\frac{1}{1-q}\right|\right) + 1 \tag{16}$$

where int(x) stands for the largest integer $\leqslant x$. It means that $\cos_q x$ has no roots for $q \leqslant 0$ or $q \geqslant 2$; $\sin_q x$ presents only one root ($x = 0$) for $q \leqslant 0.5$ or $q \geqslant 1.5$. Within these ranges, $\cos_q x$ and $\sin_q x$ present a finite number of roots (infinite number of roots occurs only for $q = 1$).

It is straightforward to show that $\phi_q(x) = \exp_q(\mathrm{i}kx)$ is an *exact* solution of the following nonlinear oscillator differential equation

$$\frac{\mathrm{d}^2[\phi(x)]^\nu}{\mathrm{d}x^2} + \gamma^2[\phi(x)]^\mu = 0 \tag{17}$$

with

$$q = \frac{\mu - \nu}{2} + 1 \qquad k^2 = \frac{2\gamma^2}{\nu(\mu+\nu)}. \tag{18}$$

When $q \to 1$, we recover the simple harmonic oscillator. It is worth stressing that $\cos_q x$ and $\sin_q x$, taken individually, are *not* solutions of equation (17), but only if combined as equation (6).

If we take into account the fact that $(\exp_q x)^a = \exp_{1-(1-q)/a}(ax)$, and $\mathrm{d}\exp_q x/\mathrm{d}x = (\exp_q x)^q$, together with equation (6), the derivatives of $\cos_q x$ and $\sin_q x$ may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos_q x = -\sin_{2-1/q}(qx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin_q x = \cos_{2-1/q}(qx). \tag{19}$$

We also have the generalization of the De Moivre theorem [30]:

$$(\cos_q x \pm \mathrm{i}\sin_q x)^a = \cos_{1-(1-q)/a}(ax) \pm \mathrm{i}\sin_{1-(1-q)/a}(ax). \tag{20}$$

We are now going to express the q-Euler formula for a complex number $z = x + \mathrm{i}y$. In order to simplify the equations, let us introduce the function $\zeta_q \equiv \ln_1 \mathrm{e}_q^z$ which satisfies $\zeta_1 = z$. If we take the 1-exponential on both sides, we may express the generalized Euler formula of a complex number z as:

$$\exp_q z = (\exp_1 \chi_q)(\cos_1 \psi_q + \mathrm{i}\sin_1 \psi_q) \tag{21}$$

where χ_q and ψ_q are defined in such a way that $\zeta_q = \chi_q + \mathrm{i}\psi_q$, that is

$$\chi_q \equiv \frac{\ln_1|\omega_q|}{1-q} \qquad \psi_q \equiv \frac{\arg(\omega_q)}{1-q} \qquad -\pi < (1-q)\psi_q \leqslant \pi \tag{22}$$

with $\omega_q = 1 + (1-q)z$.

Another way to express the q-exponential of a complex number is

$$\exp_q z = \exp_q x\left\{\cos_q\left[\frac{y}{1+(1-q)x}\right] + \mathrm{i}\sin_q\left[\frac{y}{1+(1-q)x}\right]\right\}. \tag{23}$$

This expression is valid provided that $\exp_q x$ is real and $\forall x \neq (q-1)^{-1}$. This happens for $\mathrm{Re}(\omega_q) > 0$, or for integer values of $1/(1-q)$. Equations (21) and (23) are the q-generalized Euler formula for complex numbers. Equating one another, it results in

$$(\exp_q x)\cos_q\left[\frac{y}{1+(1-q)x}\right] = (\exp_1 \chi_q)\cos_1 \psi_q \tag{24}$$

$$(\exp_q x)\sin_q\left[\frac{y}{1+(1-q)x}\right] = (\exp_1 \chi_q)\sin_1 \psi_q. \tag{25}$$

Dividing (25) by (24), we find

$$\tan_q\left[\frac{y}{1+(1-q)x}\right]=\tan_1\psi_q. \qquad (26)$$

Equations (6), (9) and (11) are particular cases of equations (23)–(26) respectively, for a pure imaginary number iy where $\exp_1\chi_q/\exp_q x$ is the general form of $\rho_q(x)$, and ψ_q is that of $\varphi_q(x)$ (equations (10)).

The comparison of equation (21) with the ordinary Euler formula $\mathrm{e}_1^z = \mathrm{e}_1^x(\cos_1 y + \mathrm{i}\sin_1 y)$ gives us an interesting remark: both e_1^z and e_q^z may be split into two factors, one responsible for the amplitude and the other responsible for the oscillations. In ordinary ($q = 1$) functions, the real and imaginary parts of a complex number are decoupled, so to say, whereas $q \neq 1$ introduces a kind of *coupling* between x and y, and both the amplitude and the oscillator factors depend on both real and imaginary parts of z.

3. Generalized q-Hyperbolic functions

We are naturally tempted to extend these ideas to hyperbolic functions. So, let us assume by definition

$$\cosh_q x \equiv \frac{\exp_q(x)+\exp_q(-x)}{2} \qquad \sinh_q x \equiv \frac{\exp_q(x)-\exp_q(-x)}{2}. \qquad (27)$$

These definitions lead us to the following relation:

$$\cosh_q^2 x - \sinh_q^2 x = \exp_q(x)\exp_q(-x) = \exp_q[-(1-q)x^2]\cdot \qquad (28)$$

The De Moivre theorem for the q-hyperbolic functions is given by

$$(\cosh_q x + \sinh_q x)^a = \cosh_{1-(1-q)/a}(ax) + \sinh_{1-(1-q)/a}(ax) \qquad (29)$$

and the derivatives of the q-hyperbolic functions are

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh_q x = \sinh_{2-1/q}(qx) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh_q x = \cosh_{2-1/q}(qx)\cdot \qquad (30)$$

The connection between the usual circular and hyperbolic functions is established by the definition of such functions of complex numbers. Here we are going to proceed in a similar way, and we straightforwardly find:

$$\begin{aligned}
\cosh_q z = {} & \frac{1}{2}\cosh_q x\left\{\cos_q\left[\frac{y}{1-(1-q)x}\right]+\cos_q\left[\frac{y}{1+(1-q)x}\right]\right\}\\
&+\frac{1}{2}\mathrm{i}\sinh_q x\left\{\sin_q\left[\frac{y}{1-(1-q)x}\right]+\sin_q\left[\frac{y}{1+(1-q)x}\right]\right\}\\
&-\frac{1}{2}\sinh_q x\left\{\cos_q\left[\frac{y}{1-(1-q)x}\right]-\cos_q\left[\frac{y}{1+(1-q)x}\right]\right\}\\
&-\frac{1}{2}\mathrm{i}\cosh_q x\left\{\sin_q\left[\frac{y}{1-(1-q)x}\right]-\sin_q\left[\frac{y}{1+(1-q)x}\right]\right\} \qquad (31)\\
\sinh_q z = {} & \frac{1}{2}\sinh_q x\left\{\cos_q\left[\frac{y}{1-(1-q)x}\right]+\cos_q\left[\frac{y}{1+(1-q)x}\right]\right\}\\
&+\frac{1}{2}\mathrm{i}\cosh_q x\left\{\sin_q\left[\frac{y}{1-(1-q)x}\right]+\sin_q\left[\frac{y}{1+(1-q)x}\right]\right\}\\
&-\frac{1}{2}\cosh_q x\left\{\cos_q\left[\frac{y}{1-(1-q)x}\right]-\cos_q\left[\frac{y}{1+(1-q)x}\right]\right\}
\end{aligned}$$

$$-\frac{1}{2}\mathrm{i}\sinh_q x\left\{\sin_q\left[\frac{y}{1-(1-q)x}\right]-\sin_q\left[\frac{y}{1+(1-q)x}\right]\right\} \quad (32)$$

with $x \neq |1-q|^{-1}$.

4. Conclusions

We have developed a generalization of the usual circular and hyperbolic functions, based on a q-exponential suggested by the Tsallis formalism of statistical mechanics. Such a generalization is a consistent q-deformation of the logarithmic and exponential functions.

We have established some basic relations for the proposed q-trigonometry, for example, the Euler formula, the Pythagoras theorem, the De Moivre theorem, the relation between q-circular and q-hyperbolic functions. These relations keep a close analogy with the usual ones and are reduced to them in the $q \to 1$ limit.

The q-circular functions present oscillatory behaviour only within a range of values of q ($0 < q < 2$ for the q-cosine and $0.5 < q < 1.5$ for the q-sine). The number of roots of these functions is finite, except if $q = 1$, when they present an infinite number of roots.

We found that $\phi_q(x) = \exp_q(\mathrm{i}kx)$ is an exact solution of the nonlinear oscillator $[\phi^\nu]'' + \gamma^2\phi^\mu = 0$, where q and k are functions of μ, ν and γ. The oscillations damp for $\mu > \nu$ ($q > 1$) and diverge for $\mu < \nu$ ($q < 1$), when $|x| \to \infty$.

The generalized Euler formula may be given by a product of an amplitude factor and an oscillatory factor, but, in contrast to the usual Euler formula, *both* the amplitude and oscillatory factors of e_q^z depend on *both* the real and imaginary parts of z.

Hopefully, the present generalization of the circular and hyperbolic functions, as well as their associated properties, can play a useful role in the actively studied Tsallis statistics.

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